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A. I. Kalandiya

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## ON CONTACT PROBLEMS OF ELASTICITY THEORY

A. I. Kalandiya

This article reviews certain contact problems in the two- /389\*  
dimensional theory of elasticity when round bodies touch without friction along most of their boundary and, therefore, Herz' hypothesis on the smallness of the contact area cannot be used. It derives fundamental equations coinciding externally with the equation in the theory of a finite-span wing with unknown parameter. These equations are solved using Multhopp's well-known technique<sup>1</sup>, and numerical calculations are performed in specific examples.

1. Let a rigid die with flat symmetrical base be pressed by a force acting along the die's axis into an elastic medium which is an infinite plane with a round opening.

It is assumed that the die can move only forward and, in addition, that there are no stresses or rotation at infinity.

The shape of the die's base (close to the contour of the opening) and the main vector of external forces squeezing the die toward the medium's boundary are given.

The stressed state of the elastic body is sought.

Let this elastic medium occupy the plane of variable  $\zeta = \xi + i\eta$ , from which a circle with center at point  $\zeta = 0$  with radius 1 is removed. We will assume that a single force of magnitude  $P$ , directed opposite axis  $\eta$ , acts on the die.

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\*Numbers in the margin indicate pagination of the foreign text.

Boundary conditions for the task are written as (cf. [2], p. 429):

$$N=0, \quad T=0 \quad \text{на } L_1; \quad T=0, \quad v_p=g(\sigma) \quad \text{на } L_2 \quad (1.1)$$

whereby the second of these conditions are fulfilled at contact arc  $L_2$ , not given beforehand, while the first is fulfilled at the rest of the circumference  $L$ ,  $L=L_1+L_2$ . Here,  $N$  and  $T$  are, respectively, the normal and tangential components of external stress acting on contour  $L$ ,  $v_p$  is normal (elastic) shift;  $g(\sigma)$ , given at  $L_2$ , is a true function of point  $\sigma=e^{ia}$ , which describes the shape of the die's base and, by virtue of symmetry, satisfies the condition  $g(\sigma)=g(-\bar{\sigma})$ .

Henceforth, we will assume that  $g(\sigma)$  has a second derivative along the arc of the contour which satisfied Gel'der's equation.<sup>1</sup>

To solve the problem, we will introduce Kolosov-Muskhelishvili's functions  $\phi(\zeta)$  and  $\psi(\zeta)$ . As we know, the following relation exists at the boundary of the area ([2], p. 335).

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$$\Phi(\sigma) + \overline{\Phi(\sigma)} - \sigma\Phi'(\sigma) - \sigma^2\Psi'(\sigma) = N - iT \quad (1.2)$$

According to the condition accepted above, at infinity we will have the following if  $|\zeta|$  is large

$$\Phi(\zeta) = \frac{a}{\zeta} + O(\zeta^{-2}), \quad \Psi(\zeta) = \frac{b}{\zeta} + O(\zeta^{-2}) \quad (1.3)$$

whereby coefficients  $A$  and  $b$  are expressed as follows by components of the main vector  $(0, -P)$  of external forces;

$$a = \frac{iP}{2\pi(1+\nu)}, \quad b = \frac{i\nu P}{2\pi(1+\nu)} \quad (\nu = 3-4\nu) \quad (1.4)$$

Here  $\nu$  is Poisson's coefficient.

The condition  $T=0$  at  $L$  by virtue of (1.2) is presented as

$$\sigma\Phi'(\sigma) + \sigma^2\Psi(\sigma) - \overline{\sigma\Phi'(\sigma)} - \overline{\sigma^2\Psi(\sigma)} = 0 \quad (1.5)$$

Hence, performing the operation

$$\frac{1}{2\pi i} \int_L \frac{dz}{z-\zeta}$$

and taking (1.3) into account, we immediately find

$$\begin{aligned} \zeta\Phi'(\zeta) + \zeta^2\Psi(\zeta) &= b(\zeta - \zeta^{-1}) + A \quad \text{at } |\zeta| \geq 1 \\ A &= \lim_{\zeta \rightarrow \infty} \zeta^2 [\Psi(\zeta) - b\zeta^{-1}] \quad \text{at } \zeta \rightarrow \infty \end{aligned} \quad (1.6)$$

To determine constant  $A$ , we will multiply (1.2) times  $\sigma^{-1}d$  and integrate the resulting equality along  $L$ . We will have

$$A = -\frac{1}{2\pi i} \int_{L_1} \frac{N(\sigma) d\sigma}{\sigma} \quad (1.7)$$

On the basis of (1.6), equality (1.2) takes the form:

$$\Phi(z) + \overline{\Phi(\bar{z})} - b(z - \bar{z}) - A = N \quad \text{at } L$$

Hence, as above, we find

$$\Phi(\zeta) = -\frac{1}{2\pi i} \int_{L_1} \frac{N(\sigma) d\sigma}{\sigma - \zeta} - \frac{b}{\zeta} \quad \text{at } |\zeta| > 1 \quad (1.8)$$

Differentiating (1.8) in terms of  $\zeta$  and taking into account the continuity<sup>2</sup> of normal stress  $n(\sigma)$  at boundary  $L$ , we obtain

$$\Phi'(\zeta) = -\frac{1}{2\pi i} \int_{L_1} \frac{N'(\sigma) d\sigma}{\sigma - \zeta} + \frac{b}{\zeta^2} \quad (1.9)$$

To determine  $N(\sigma)$  at  $L$ , we will use the well-known formula ([2], p. 135)

$$2\mu v_p = \operatorname{Re} \{ \bar{\sigma} [z\zeta'(\sigma) - \sigma\zeta'(\sigma) - \psi(\sigma)] \} \quad (1.10)$$

If we take (1.5) into account, we obtain the following equation from this equality by differentiating in terms of  $\alpha$   $z = e^{i\alpha}$ ,

$$2\alpha \left[ r_f + \frac{d^2 r_f}{dx^2} \right] = \operatorname{Re} \{ (z-1)\Phi(z) - (z+1)\sigma\Phi'(z) + \sigma\Phi'(z) + z^2\Phi'(z) \} \text{ на } L_2 \quad (1.11)$$

Introducing here the limit values for the functions  $\phi(\zeta)$ ,  $\phi'(\zeta)$  given by equations (1.8) and (1.9) and the expression for the left side of (1.6), on the basis of the latter, from condition (1.1) we obtain the equation

$$\begin{aligned} \frac{z-1}{z+1} N(\sigma_0) + \frac{\sigma_0}{\pi i} \int_{L_2} \frac{N'(z) dz}{z-\sigma_0} - \frac{1}{2\pi i} \int_{L_2} \frac{N'(z) dz}{z} + \frac{\alpha P i}{\pi(1+\alpha)} (\sigma_0 - \bar{\sigma}_0) &= \\ &= \frac{4\alpha}{z+1} \left[ g(z) + \frac{d^2 g(z)}{dx^2} \right] \end{aligned} \quad (1.12)$$

To determine the integration line in this singular integral-differential equation, we have one more relation

$$\int_{L_2} N(z) dz = -P \quad (1.13)$$

After finding  $N(\sigma)$  and  $L_2$ , we will determine stress functions with (1.8) and (1.6). In the particular case when the rigid die is a round washer inserted into an opening of the same radius, we will have  $g(\sigma) = 0$  at  $L_2$ , and equation (1.2) will take the form:

$$\frac{z-1}{z+1} N(\sigma_0) + \frac{\sigma_0}{\pi i} \int_{L_2} \frac{N'(z) dz}{z-\sigma_0} - \frac{1}{2\pi i} \int_{L_2} \frac{N'(z) dz}{z} = \frac{\alpha P}{\pi i(1+\alpha)} (\sigma_0 - \bar{\sigma}_0) \quad (1.4)$$

Note. Equation (1.12) (or, more precisely, the equation derived from (1.12) when its right side is replaced with a certain approximate expression) is found in V. V. Panasyuk's article [3].

This equation is derived somewhat differently on the basis of the results of I. N. Kartsvadze (e.g. [2], paragraphs 125, 126). The author apparently was not familiar with relatively

new analytical tools recently developed to solve this integral-differential equation (cf. below, paragraph 3).

Note also that the case of a round die (of the same radius as the opening) was considered by M. P. Sheremet'yev [4], who proposed a method for constructing equation (1.14) somewhat different than that used here.

However, the author's reasoning contains an omission because of which the equation derived in [4] is incorrect.

As we know, in the two-dimensional deformed case, deformation  $e_{xx}$  and stresses  $\sigma_\rho, \sigma_\alpha$  in polar coordinates are, according to Hook's law, related as follows

$$e_{xx} = \frac{1}{E} (\sigma_x - \nu\sigma_\rho) - \frac{\nu^2}{E} (\sigma_\rho + \sigma_\alpha) \quad (1.15)$$

where E is the modulus of elasticity. A similar formula suitable for the case of a generalized two-dimensional stressed state, takes the form:

$$e_{xx} = \frac{1}{E} (\sigma_x - \nu\sigma_\rho) \quad (1.16)$$

M. P. Sheremet'yev [4] uses only equation (1.16) ([4], p. 439, equation (1.10)), in view of which his reasoning is sound only in the second of the two-dimensional cases. In this case, the author's equation [p. 441, (2.15)] indeed coincides with (1.14) (only this time one must take  $\alpha = (3-\nu)/(1+\nu)$ ). One must take into account that, in reference [5], force P is directed along axis  $\xi$ .

This note pertains also to the author's reasoning in other cases of the equilibrium of an infinite plane which were considered in the same work [4].

2. Let a round elastic washer, generally with other elastic properties, be inserted into a round opening in an infinite medium, the opening and washer having the same radius. It is assumed that the washer is squeezed toward the surrounding material by a concentrated force applied to its center. (As before, we will assume that stresses and rotation disappear at infinity.)

All elements relating to the washer (elastic constants, stress functions, etc.) are symbolized by 0, and we will write contour conditions of the task in the form (cf. [2], p. 207)

$$\begin{aligned} N=0, \quad T=0 \quad \text{на } L_1, \quad T=0, \quad v_p^0 = v_p \\ N_0 - iT_0 = N - iT \quad \text{на } L \end{aligned} \quad (2.1)$$

The stress functions  $\phi(\zeta)$  and  $\psi(\zeta)$ , which correspond to an infinite plate, will, as before, be expressed by equations (1.8) and (1.6).

Functions  $\phi_0(\zeta)$  and  $\psi_0(\zeta)$ , which correspond to an elastic washer, will obviously take the form:

$$\begin{aligned} \Phi_0(\zeta) = \frac{a_0}{\zeta} + \Phi_0^*(\zeta), \quad \Psi_0(\zeta) = \frac{b_0}{\zeta} + \Psi_0^*(\zeta) \\ \left( a_0 = \frac{iP}{2\pi(1+\nu_0)}, \quad b_0 = a_0\nu_0 \right) \end{aligned} \quad (2.2)$$

whereby functions  $\Phi_0^*(\zeta)$  and  $\Psi_0^*(\zeta)$  are holomorphs in a unitary circle.

From the appropriate conditions (2.1), we obtain the following, which is entirely similar to the previous

$$\zeta\Phi_0'(\zeta) + \zeta^2\Psi_0'(\zeta) = a_0(\zeta - \zeta^{-1}), \quad |\zeta| \leq 1 \quad (2.3)$$

$$\Phi_0(\zeta) = \frac{1}{2\pi i} \int_L \frac{N(\sigma) d\sigma}{\sigma - \zeta} + a_0 \left( 2\zeta + \frac{1}{\zeta} \right) + \frac{A}{2}, \quad |\zeta| < 1 \quad (2.4)$$

where constant A is given by equation (1.7); it is assumed that

$\text{Im} \phi_0^*(0) = 0$ . From (2.4), using, as before, the condition

$$N(\sigma_*) = N(-\bar{\sigma}_*) = 0 \quad (2.5)$$

$\sigma_*$ ,  $-\bar{\sigma}_*$  are end points of arc  $L_2$ , we find

$$\Phi_0'(\zeta) = \frac{1}{2\pi i} \int_{L_2} \frac{N'(z) dz}{z - \zeta} + a_0 \left(2 - \frac{1}{\zeta^2}\right) \quad (2.6)$$

Substituting the left side for  $v_p^0$  and  $v_p$  and expressing the equality

$$v_p^0 + \frac{d^2 v_p^0}{dz^2} = v_p + \frac{d^2 v_p}{dz^2}$$

we have

$$\begin{aligned} & \mu_0 \text{Re} \{ (z-1) \Phi(z) - (z+1) \sigma \Phi'(z) + \sigma \Phi'(z) + \sigma^2 \Psi(z) \} = \\ & = \mu \text{Re} \{ (z_0-1) \Phi_0(z) - (z_0+1) \sigma \Phi_0'(z) + \sigma \Phi_0'(z) + \sigma^2 \Psi_0(z) \} \end{aligned} \quad (2.7)$$

Introducing the corresponding expressions given by the previous formulas, after certain simplification we obtain

$$kN(\sigma_0) + \frac{\sigma_0}{\pi i} \int_{L_1} \frac{N'(z) dz}{z - \sigma_0} - \frac{p}{2\pi i} \int_{L_2} \frac{N(z) dz}{z} = \frac{q}{2\pi i} P(\sigma_0 - \bar{\sigma}_0) \quad (2.8)$$

where

$$k = \frac{1}{2} \frac{(1-2\nu)(1+\nu)E_0 - (1-2\nu_0)(1+\nu_0)E}{(1-\nu^2)E_0 + (1-\nu_0^2)E} \quad (2.9)$$

$$p = \frac{(1-\nu^2)E_0}{(1-\nu^2)E_0 + (1-\nu_0^2)E}, \quad q = \frac{1}{2} \frac{(1+\nu)x E_0 + (1+\nu_0)E}{(1-\nu^2)E_0 + (1-\nu_0^2)E} \quad (2.10)$$

whereby  $E$  represents the modulus of elasticity. The contact arc  $L_2$  is, as before, unknown, by virtue of which relation (1.13) should be attached to equation (2.8) as before.

When the washer and the plate are made of the same material ( $\nu_0 = \nu$ ,  $E_0 = E$ ), coefficient  $k$  in (2.8) becomes zero, as a result of which this equation is solved in closed form. The solution (closed) to the problem in this case was found in [5].

Moving in equation (2.6) to the limit at  $E_0 \rightarrow \infty$ , we come to the case of an absolutely rigid washer -- equation (1.14).

If  $E = \infty$ , we have the case of a round, elastic washer inserted into an opening of equal radius in an absolutely rigid plate pressed toward it by a concentrated force applied to the center of the washer. Equation (2.8) gives

$$-\frac{x-1}{x+1}N(\sigma_0) + \frac{\sigma_0}{\pi i} \int_L \frac{N'(\sigma) d\sigma}{\sigma - \sigma_0} = \frac{P}{\pi i(1+x)}(\sigma_0 - \bar{\sigma}_0) \quad (2.11)$$

whereby  $\lambda$  is the washer's elastic constant.

Equation (2.8) is an example of an equation for a two-dimensional contact problem covered without any limiting assumptions.<sup>3</sup>

3. Let us add to equation (2.8) a new variable introduced by the relation

$$\sigma = i \frac{x - i\beta}{x + i\beta}, \quad \beta = \frac{\cos \alpha_*}{1 + \sin \alpha_*}, \quad \sigma_* = e^{i\alpha_*} \quad (3.1)$$

The previous function, as we know, gives the conversion of the unitary circumference  $|\sigma| = 1$  into real axis  $x$ , transferring arc  $L_2$  to segment  $[-1, 1]$ ; the point  $-\bar{\sigma}_*$ ,  $\sigma_*$  on the circumference is transferred<sup>4</sup> respectively to points  $x = -1$ ,  $x = 1$ .

After the obvious transformations, we will have

$$\frac{k\beta}{x^2 + \beta^2} N(x) - \frac{1}{2\pi} \int_{-1}^1 \frac{N'(t) dt}{t-x} - \frac{\beta^2}{\pi(x^2 + \beta^2)} \int_{-1}^1 \frac{N(t) dt}{t^2 + \beta^2} = \frac{q\beta}{\pi} P \frac{x^2 - \beta^2}{(x^2 + \beta^2)^2} \quad (3.2)$$

$$N(1) = N(-1) = 0$$

whereby, for simplicity, we again assume  $N(\sigma) = N(x)$ . The equality (1.3) is transformed to

$$2\beta \int_{-1}^1 \frac{t^2 - \beta^2}{(t^2 + \beta^2)^2} N(t) dt = P \quad (3.3)$$

If we disregard unknown parameter  $\beta$ , equation (3.2) is /394  
the familiar equation of the theory of a finite-span wing, which  
is usually written as:

$$\frac{\Gamma(x)}{B(x)} - \frac{1}{2\pi} \int_{-1}^1 \frac{\Gamma(t) dt}{t-x} = f(x) \quad (3.4)$$

Here B and f are assigned functions at segment [-1, 1],  
wherein B(x) is zero nowhere except the ends of this segment,  
while  $\Gamma(x)$  is the desired function subordinate to condition  
 $\Gamma(1) = \Gamma(-1) = 0$ .

Many articles have been devoted to the wing theory equation  
(cf. e.g. [7, 8]). Among the countless methods of numerical  
solution devoted to this equation, the most successful from the  
standpoint of practical applications is Multhopp's (direct)  
method [1], which, because of its simplicity and efficiency, is  
still considered the best mathematical device for aerodynamic  
calculation of a wing.

According to those methods, the desired  $\Gamma$  takes the form of  
a trigonometric interpolation polynomial

$$\begin{aligned} L[\Gamma; x] &= \sum_{k=1}^n (-1)^{k+1} \Gamma(x_k) \frac{\sin(n+1)\vartheta \sin\vartheta_k}{\cos\vartheta - \cos\vartheta_k} = \\ &= \frac{2}{n+1} \sum_{k=1}^n \Gamma(x_k) \sum_{m=1}^n \sin m\vartheta_k \sin m\vartheta \end{aligned} \quad (3.5)$$

$$x = \cos\vartheta, \quad x_k = \cos\vartheta_k, \quad \vartheta_k = \frac{k\pi}{n+1} \quad (0 \leq \vartheta \leq \pi; k = 1, \dots, n)$$

and, after application of a certain squaring formula to a  
special integral on the left side of (3.4), this equation is  
replaced by a system of linear equations of relatively  
approximate values  $\bar{\Gamma}_k$  of the desired function in the given  
(Chebyshev) nodes  $x_k$ . This system takes the form

$$\left(\frac{1}{B_m} + b_{mm}\right) \bar{\Gamma}_m = f_m + \sum_{k=1}^n b_{mk} \bar{\Gamma}_k \quad (m = 1, \dots, n) \quad (3.6)$$

where  $B_m + B(x_m)$ ,  $f_m = f(x_m)$ ;  $b_{mk}$  is also known, so that  $b_{mk} = 0$  at  $m-k = 2, 4, \dots$ . Substituting the solution to this system into the right side of (3.5) instead of  $\Gamma_k$ , we obtain the approximate solution to the equation (3.4).

Multhopp's article [1] proved that consecutive approximation method for (3.6) always converges if  $B(x)$  is negative [1]. With negative  $B(x)$ , which occurs (as can be seen from equation (3.2)) in applications to elasticity theory, iteration method as applied to system (3.6) also yields a converging process, provided that  $B(x)$  satisfied the condition<sup>6</sup>

$$\max_{x \in [-1, 1]} \left| \frac{1}{B(x)} \right| < \frac{1}{\varepsilon} \quad (3.7)$$

We will apply this method to solving the equations derived /395 here.

Solving (3.2) and (3.3) together, we will, according to (3.5), find the approximate solution to (3.2) in the form

$$N(x) = \frac{2\sqrt{1-x^2}}{n+1} \sum_{m=1}^n a_m U_{m-1}(x), \quad a_m = \sum_{k=1}^n \bar{N}_k \sin m\theta_k \quad (3.8)$$

$$U_{m-1}(x) = \frac{\sin m\theta}{\sin \theta}$$

The function  $U_{m-1}(x)$  is a polynomial of the  $x$ -th order of  $m-1$  (a second-order Chebyshev polynomial).

To calculate the desired values for the problem, it is convenient to shift from the physical plane of variable  $\zeta$  to the plane of variable  $z = x + iy$  with the relation

$$\zeta = i \frac{z - i\beta}{z + i\beta} \quad \left( \beta = \frac{\cos \alpha_*}{1 + \sin \alpha_*} \right) \quad (3.9)$$

which accomplishes the conformal transformation of the circle  $|\zeta| < 1$  to the upper half-plane  $\text{Im } z > 0$ . After transformations, we obtain the formula

$$\frac{1}{2\pi i} \int_{L_2} \frac{N(z) dz}{z-z} = \frac{1}{2\pi i} \int_{-1}^1 \frac{N(x) dx}{x-z} + \frac{\beta}{2\pi} \int_{-1}^1 \frac{N(x) dx}{x^2 + \beta^2} \quad (3.10)$$

But the Koche integral on the right side of (3.10) is calculated in final form if  $N(x)$  takes the form of (3.8). Consequently, after Multhopp's method is used to solve the equation in the contact problem, the corresponding functions for stress are determined in closed form.

**Example 1.** An absolutely rigid round washer squeezed against an elastic plate (external problem). The problem's equation is derived from (1.1) by substituting (3.1) or the limit transition in (3.2) at  $E_0 \rightarrow \infty$ . We will write this equation as

$$\frac{x-1}{x+1} \frac{\beta}{x^2 + \beta^2} N(x) - \frac{1}{2\pi} \int_{-1}^1 \frac{N'(t) dt}{t-x} = \frac{\beta^2 q}{x^2 + \beta^2} + \frac{2x\beta P}{\pi(1+x)} \frac{x^2 - \beta^2}{(x^2 + \beta^2)^2} \quad (3.11)$$

where

$$q = \frac{1}{\pi} \int_{-1}^1 \frac{N'(t) dt}{t^2 + \beta^2} \quad (3.12)$$

Let us remember that parameter  $\beta$ , introduced into (3.11) and characterizing the size of the contact area, is unknown and, in addition, the right side of the equation contains the still undefined constant  $q$  in the form of functional (3.12). In the rest, (3.11) coincides with (3.4), so that  $B(x) > 0$  at  $[-1, 1]$ .

Equality (3.3) is rewritten as:

$$c(\beta) \equiv \frac{2\beta}{P} \int_{-1}^1 \frac{t^2 - \beta^2}{(t^2 + \beta^2)^2} N'(t) dt = 1 \quad (3.13)$$

In our (symmetric) case of equation (3.11), if  $n$  is odd, system (3.6) will take the form:

$$\begin{aligned} & \left[ \frac{x-1}{x+1} \frac{\beta}{\cos^2 \theta_m + \beta^2} + b_{mm} \right] \bar{N}_m = \\ & = \sum_{k=1}^{1/2(n+1)} B_{mk} \bar{N}_k + \frac{\beta^2 q}{\cos^2 \theta_m + \beta^2} + \frac{2x\beta^3}{\pi(1+x)} \frac{\cos^2 \theta_m - \beta^2}{(\cos^2 \theta_m + \beta^2)^2} \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} \bar{\Gamma}_k &= \bar{\Gamma}_{n+1-k} & (m = 1, 2, \dots, 1/2(n+1)) \\ B_{mk} &= b_{mk} + b_{m, n+1-k} & (k = 1, \dots, 1/2(n-1), B_{m, 1/2(n+1)} = b_{m, 1/2(n+1)}) \end{aligned} \quad (3.15)$$

Solving this system for a certain parameter  $\beta$  (e.g. at  $\beta=1$ ), we will find unknown  $\bar{N}_k$  in the form /396

$$\bar{N}_k = N_k^{(0)} q + N_k^{(1)} \quad (k = 1, \dots, n) \quad (3.16)$$

whereby  $N_k^{(0)}$ ,  $N_k^{(1)}$  will be known. The corresponding approximate solution to  $\bar{N}(x)$  will obviously contain constant  $q$ , defined as a result of (3.12). Then, after  $\bar{N}(x)$  is calculated, the left side of equality (3.13) is determined. To determine  $q$  and  $c(\beta)$ , a Gaussian squaring formula is used (e.g. [9], p. 617).

$$\begin{aligned} q &= \frac{1}{n+1} \sum_{k=1}^n \frac{\sin \vartheta_k \bar{N}_k}{\cos^2 \vartheta_k + \beta^2} \\ c(\beta) &= \frac{2\beta\pi}{n+1} \sum_{k=1}^n \frac{\sin \vartheta_k (\cos^2 \vartheta_k - \beta^2) \bar{N}_k}{(\cos^2 \vartheta_k + \beta^2)^2} \frac{1}{P} = 1 \end{aligned} \quad (3.17)$$

The value of  $\beta$  which we have taken and the  $N(x)$  corresponding to it will generally not satisfy equality (3.13). Therefore, selecting new  $\beta$  values, we will repeat our calculations until this equality is satisfied with necessary accuracy. As a result, we will have certain approximate values for  $\beta$  and  $N(x)$  for a given  $n$ . Then other  $n$ 's are taken and these calculations repeated.

In system (3.14), we will first assume  $n=7$  and will set Poisson's coefficient equal to  $1/3$  for further calculations. In expanded form the resulting system appears as:<sup>7</sup>

$$\begin{aligned} & \left[ \frac{\beta}{4(\cos^2 \vartheta_1 + \beta^2)} + 5.2262 \right] \bar{N}_1 = \\ & = 1.9142 \bar{N}_2 + 0.1464 \bar{N}_4 + \frac{q\beta^2}{\cos^2 \vartheta_1 + \beta^2} + \frac{5}{4} \beta \frac{\cos^2 \vartheta_1 - \beta^2}{(\cos^2 \vartheta_1 + \beta^2)^2} \frac{P}{\pi} \end{aligned}$$

$$\begin{aligned}
& \left[ \frac{\beta}{4(\cos^2 \vartheta_3 + \beta^2)} + 2.1648 \right] \bar{N}_3 = \\
& = 0.9142 \bar{N}_2 + 0.8536 \bar{N}_4 + \frac{q\beta^2}{\cos^2 \vartheta_3 + \beta^2} + \frac{5}{4} \beta \frac{\cos^2 \vartheta_3 - \beta^2}{(\cos^2 \vartheta_3 + \beta^2)^2} \frac{P}{\pi} \\
& \left[ \frac{\beta}{4(\cos^2 \vartheta_2 + \beta^2)} + 2.8284 \right] \bar{N}_2 = \\
& = 1.0360 \bar{N}_1 + 1.1944 \bar{N}_3 + \frac{q\beta^2}{\cos^2 \vartheta_2 + \beta^2} + \frac{5}{4} \beta \frac{\cos^2 \vartheta_2 - \beta^2}{(\cos^2 \vartheta_2 + \beta^2)^2} \frac{P}{\pi} \\
& \left[ \frac{\beta}{4(\cos^2 \vartheta_4 + \beta^2)} + 2.0000 \right] \bar{N}_4 = \\
& = 0.1121 \bar{N}_1 + 1.5774 \bar{N}_3 + \frac{q\beta^2}{\cos^2 \vartheta_4 + \beta^2} + \frac{5}{4} \beta \frac{\cos^2 \vartheta_4 - \beta^2}{(\cos^2 \vartheta_4 + \beta^2)^2} \frac{P}{\pi} \\
& \vartheta_k = \frac{k\pi}{8}, \quad k = 1, 2, 3, 4 \tag{3.18}
\end{aligned}$$

Solving this system together with (3.17), we will have (values for  $q$  and  $N_k$  [1] in (3.16) as well as  $\bar{N}_k$  are given in  $P/\pi$ )<sup>8</sup>

$$\beta = 1.20886, \quad q = -0.52422 \tag{3.19}$$

$$\begin{aligned}
\bar{N}_1 &= 0.48035 q - 0.29067 = -0.54248, \quad \bar{N}_2 = 0.91050 q - 0.63176 = -1.10906 \\
\bar{N}_3 &= 1.23317 q - 1.00926 = -1.65571, \quad \bar{N}_4 = 1.35900 q - 1.20474 = -1.91715 \tag{3.20}
\end{aligned}$$

The value of the following polar angle corresponds to the 397  $\beta$  which has been found

$$\alpha_* = -10^\circ 48' 12'' \tag{3.21}$$

with which contact arc  $L_2$  is calculated. Given these values for the unknowns, the left side of (3.17) equals

$$c(\beta) = 0.99947$$

For maximum pressure in our approximation,<sup>9</sup> according to (3.20) we will have

$$N|_{r=0} \approx \bar{N}_4 = -0.61025 P \tag{3.22}$$

Later in (3.14) we<sup>\*</sup> will use  $n=15$ . This system of eight

equations (with five separate unknowns in each) will be solved by consecutive approximation method, wherein we will use the values in (3.20) with unknown  $q$  as the zero approximation of  $N_2, N_4, N_6, N_8$ .

Solving this system with the  $\beta$  found together with (3.17), we obtain Multhopp's second approximation

$$q = -0.52418 \quad (3.23)$$

$$\begin{aligned} \bar{N}_1 &= 0.24397 q - 0.14220 = -0.27008, & \bar{N}_2 &= 0.48054 q - 0.29080 = -0.54269 \\ \bar{N}_3 &= 0.70465 q - 0.45269 = -0.82206, & \bar{N}_4 &= 0.91061 q - 0.63180 = -1.10913 \\ \bar{N}_5 &= 1.09038 q - 0.82382 = -1.39537, & \bar{N}_6 &= 1.23312 q - 1.00927 = -1.65565 \\ \bar{N}_7 &= 1.32624 q - 1.15058 = -1.84577, & \bar{N}_8 &= 1.35877 q - 1.20437 = -1.91663 \end{aligned} \quad (3.24)$$

For these values of  $\beta$  and  $\bar{N}_k$  ( $k=1, \dots, 8$ ), the left side of (3.17) gives

$$c(\beta) = 0.99938$$

For maximum stress, we have

$$N|_{x=0} \simeq \bar{N}_8 = -0.61008P \quad (3.25)$$

As we can see, the desired values in the first and second approximations differ little from one another.

2. A round, elastic washer, squeezed against an opening in a rigid plate (internal problem). The appropriate equation is derived from (2.11), if we introduce (3.1) into it. It takes the form:

$$\frac{x-1}{x+1} \frac{\beta}{x^2+\beta^2} N(x) - \frac{1}{2\pi} \int_{-1}^1 \frac{N'(t) dt}{t-x} = \frac{2\beta P}{\pi(1+x)} \frac{x^2-\beta^2}{(x^2+\beta^2)^2} \quad (3.26)$$

This equation (with the stipulation made above relative to  $\beta$ ) also coincides with (3.4), so that, this time,  $B(x) = 0$  at  $[-1, 1]$ .

System (3.6) for equation (3.26) externally differs little from system (3.14). It takes the form:

$$\left[ -\frac{z-1}{z+1} \frac{\beta}{\cos^2 \vartheta_m + \beta^2} + b_{mm} \right] \bar{N}_m = \sum_{k=1}^{1/2(n+1)} B_{mk} \bar{N}_k + \frac{2\beta P}{\pi(1+x)} \frac{\cos^2 \vartheta_m - \beta^2}{(\cos^2 \vartheta_m + \beta^2)^2} \quad (3.27)$$

(m = 1, ..., 1/2(n+1))

This system should be solved, as before, by satisfying equation (3.17) and simultaneously determining parameter  $\beta$ . At n=7, we will have the following values for the desired parameters ( $\bar{N}_k$  is given in P/n):

$$\beta = 0.91612 \quad (3.28)$$

$$\bar{N}_1 = -0.37480, \bar{N}_2 = -0.85751, \bar{N}_3 = -1.47396, \bar{N}_4 = -1.84453 \quad (3.29)$$

Given these values,  $c(\beta) = 1.00034$ .

From this we obtain the following for the desired value of /398 angle  $\alpha$ , which corresponds to the end of the contact arc

$$\alpha_* = 5^\circ 00' 46'' \quad (3.30)$$

and maximum pressure will equal

$$N|_{x=0} \simeq \bar{N}_4 = -0.58713P \quad (3.31)$$

In (3.27) we will then assume n=15. Solving this system with (3.28), we obtain<sup>10</sup>

$$\begin{aligned} \bar{N}_1 &= -0.17990, \bar{N}_2 = -0.37458, \bar{N}_3 = -1.15746, \bar{N}_4 = -1.47378 \\ \bar{N}_5 &= -0.59686, \bar{N}_6 = -0.85727, \bar{N}_7 = -1.73637, \bar{N}_8 = -1.84215 \end{aligned} \quad (3.32)$$

Then  $c(\beta) = 0.99905$ . Specifically, for maximum normal stress in the second approximation we will have

$$N|_{x=0} \simeq \bar{N}_8 = -0.58638P$$

According to the values found for  $\bar{N}(x)$  and  $\beta$ , we can, as indicated above, calculate all other desired values for the corresponding contact problem. Specifically, it is easy to calculate normal shifts which will satisfy the contour condition of the problem with a certain error.<sup>11</sup>

In conclusion, let me note that these calculations were done at the USSR Academy of Sciences' Computing Center. I will take the opportunity to thank the directorate of the Computing Center for its assistance, as well as Ye. S. Bogomolova and T. M. Kopylova of the Computing Department for their rapid and accurate work.

## FOOTNOTES

1. The last condition of (1.1) should have been written more accurately:  $v_p = g(\sigma) + c \sin \alpha$ , where  $c$  is the forward movement of the die; but one can do without this shift since it can be eliminated by the rigid forward shift of the entire system.
2. It is assumed that the die has no angular points at the contact boundary. On this assumption  $N(\sigma)$  will be a continuous function at  $L$ , becoming zero at the ends of the contact arc.
3. Contact problems of this type, which result in equations of the same structure, were first considered apparently in I. Ya. Shtayerman's work (e.g. [8]). In this article, we direct our attention primarily to the possibility of efficiently solving these problems.
4.  $\sigma_*$  denotes the end point of arc  $L_2$  for which  $\operatorname{Re} \sigma_* > 0$ .
5. The functions  $B(x)$  are exclusively such in application to wing theory.
6. Justification of Multhopp's method and its application in problems discussed here are found in the author's dissertation ("Certain Mixed Problems of Elasticity Theory," Steklov Mathematical Institute, USSR Academy of Sciences, 1955).
7. Reference [1] has tables for calculated values of  $b_{mk}$  at  $n=7.15.31$ .
8. Here and henceforth we will require that relative error in (3.17) not exceed 0.1%.
9. The solution to  $\bar{N}(x)$  which corresponds to  $n=7$  is assumed, according to Multhopp, to be the first approximation.
10. Given our  $\mu$  and the Poisson's coefficient taken above, function  $B(x)$  in (3.26) satisfies condition (3.7). However, we must state that consecutive approximation method converges much more slowly in this case than in the previous case. For this reason, the method cited, when applied to system (3.6) given negative  $B(x)$  [even with (3.7)], may not always be preferable.
11. For example, the absolute error for  $\mu/P(v_p + d^2 v_p / da^2)$  for the external problem does not exceed 0.0072.

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